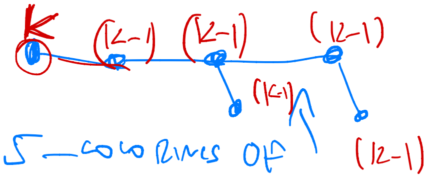
Lecture 6 - Chromatic Polynomial

For a simple graph G and an integer k, denote by $P(G,k)$ the number of k-colorings of the graph G. We call this function the *chromatic polynomial* of G.

1: For a tree T, show that $P(T,k)$ is really a polynomial.



$K.(K-1)^{n-1} = P(T,K)$

$|V(T)| = n$

2: Let G be a graph. What is the smallest k such that $P(G,k) > 0$?

$P(G, \chi(G)) \geq 1$ $P(G,K) = 0$ FOR ALL $K < \chi(G)$

The following recursion implies that G is really a polynomial. By G/e we denote the graph obtained by contracting e, i.e. identify the endpoints of e in $G - e$.

Proposition 1. *Let G be a graph and let $e = xy$ be an edge of G. Then,*

$$P(G,k) = P(G-e,k) - P(G/e,k). \tag{1}$$

3: Prove the proposition.

$P(G,K) = P(G-e,K) - P(G/e,K)$

$\varphi(u) = \varphi(v)$

IF $\varphi(u) \neq \varphi(v)$, GOOD COLORING OF G

4: Find $P(C_5, x)$ using the above recursion.

$P(C_5, x) = P(P_5, x) - P(C_4, x) = P(P_5, x) - P(P_4, x) + P(C_3, x)$

$= x.(x-1)^4 - x.(x-1)^3 + \boxed{x.(x-1).(x-2)}$

$P(C_3, x) = P(P_3, x) - P(P_2, x)$

$x.(x-1)^2 - x.(x-1) =$

$= x.(x-1).(x-1-1) =$

$= x.(x-1)(x-2)$

A *color k-partition* of G is a partition of $V(G)$ on k nonempty disjoint sets

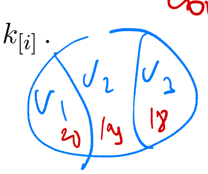
$$V_1, V_2, \ldots, V_k,$$

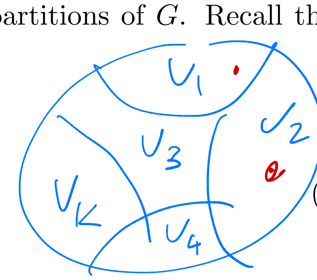
such that V_i is an independent set in G. Note that a color k-partition of G give us immediately a k-coloring of G with all V_i being its color classes. Denote by $a_k(G)$ the number of color k-partitions of G. Recall that $k_{[i]} = k(k-1)\cdots(k-i+1)$.

Proposition 2. *Let G be a graph on n vertices. Then,*

20 colors

$$P(G,k) = \sum_{i=1}^{n} a_i(G) k_{[i]}\,. \tag{2}$$

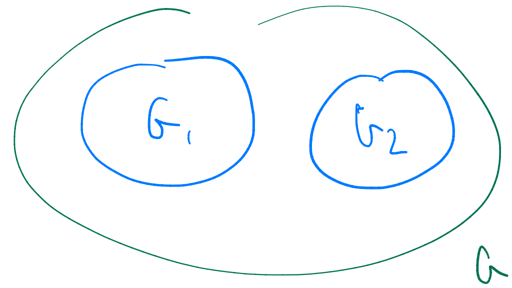




5: Prove the proposition.

Proposition 3. *Let G be disjoin union of graphs G_1 and G_2. Then,*

$$P(G,k) = P(G_1,k) \cdot P(G_2,k)\,.$$

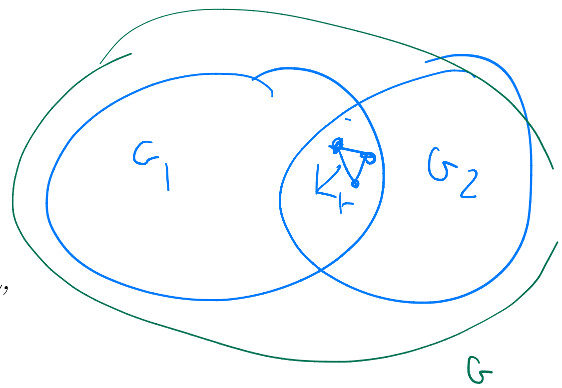


6: Prove the above proposition.

Let G be a union of G_1 and G_2 whose intersection is a clique, i.e.

$$G = G_1 \cup G_2 \quad \text{and} \quad G_1 \cap G_2 = K_r\,.$$

We say G is an r-clique-sum of G_1 and G_2.

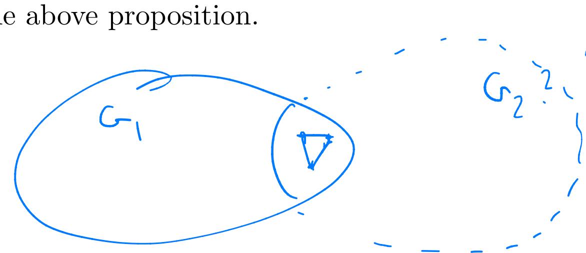


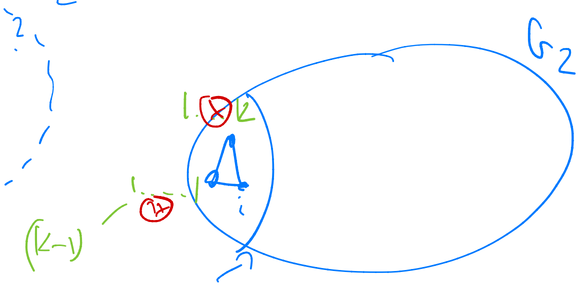
Proposition 4. *Let G be a r-clique-sum of graphs G_1 in G_2. Then,*

$$P(G,k) = \frac{P(G_1,k) \cdot P(G_2,k)}{P(K_r,k)}\,.$$

7: Prove the above proposition.

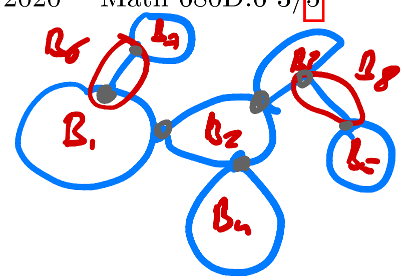
HOW MANY EXTEND ?





COLORINGS OF G_2 $P(G_2,K)$ OF THEM

$$P(K_r,K) = \frac{P(G_2,K)}{\boxed{K(K-1)(K-3)}}$$



Corollary 5. *Let G be a connected graph, whose blocks are $B_1, B_2, \ldots, B_r$. Then,*

$$P(G,k) = k^{1-r} P(B_1,k)P(B_2,k)\cdots P(B_r,k).$$

8: Prove the corollary.

INDUCTION ON r.. $G = (B_1 \ldots B_{r-1}) \cup B_r$

$$P(G,k) = \left(k^{1-(r-1)} P(B_1,k) \ldots P(B_{r-1},k)\right) \cdot \frac{P(B_r,k)}{k}$$

1 Coefficients of chromatic polynomial

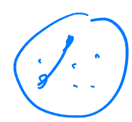
Proposition 6. *Let G be a simple graph on n vertices and with m edges. For the chromatic polynomial $P(G,k)$ the following holds:*

(a) *coefficient at k^n is 1,*

(b) *the free coefficient is 0,*

(c) *coefficient at k^{n-1} is $-m$,*

(d) *sign of the coefficients alternate.*



$P(\overline{K_{n-1}}, k)$

9: Prove the proposition. Hint: Use induction on m starting with $m = 0$.

$P(\overline{K_n}, k) = k^n$

E_n = 1 EDGE $P(E_n,k) = P(\overline{K_n},k) - P(\overline{K_{n-1}},k)$

$= k^n - k^{n-1}$

$\square$ $P(G,k) = \underbrace{\quad}_{n} - \underbrace{\quad}_{n-1} = \ldots \ldots P(\overline{K_n},k) \pm (\quad)$

$1 \cdot k^n$ AT MOST k^{n-1} n-1 OR LESS VERTICES

b) .. CANNOT REDUCE BELOW k

$$P(G,k) = \sum_{i=1}^{n} P(\overline{K_i},k)\,\alpha_i = \sum_{i=1}^{n} \alpha_i k^i + 0$$

c) $P(G,k) = k^n - m k^{n-1} + \ldots$

$= P(G-e,k) - P(G/e,k) = k^n - (m-1)k^{n-1} + \ldots - k^{n-1} + \ldots$

$= k^n - m k^{n-1} + \ldots$

d) SIGNS ALTERNATE $P(G,k) = P(G-e,k) - (P(G/e,k))$

$+k^n - k^{n-1} + k^{n-2} + \ldots$ $-(k^{n-1} - k^{n-2} + k^{n-3})$

$\binom{m}{2}-t=\binom{5}{2}-1$

10: Find chromatic polynomials of $P_3 \cup C_3$ and $K_1 \cup K_2 \cup K_3$.

$$P(P_3 \cup C_3, x) = x^6 - 5x^5 + 9x^4 - 7x^3 + 2x^2$$

$\binom{m}{2}-t = \binom{4}{2}-1$

$$(x \cdot (x-1)^2 \cdot x(x-1)(x-2))$$

$$P(K_1 \cup K_2 \cup K_3, x) = x^6 - 4x^5 + 5x^4 - 2x^3$$

$$x \cdot (x \cdot (x-1)) \cdot (x \cdot (x-1)(x-2))$$

The first one has two components and the first non-zero coefficient at the second power. Similarly the second one has three components and its first non-zero coefficient at the third power. Now we show that this holds in general.

Proposition 7. *The degree of the smallest non-zero coefficient of $P(G,k)$ equals the number of components of G.*

11: Prove the proposition. Hint: Induction on m.

$m=0 \quad P(\overline{K_n}, x) = x^n$

$m>0 \quad P(G,x) = \underbrace{P(G-e,x)}_{\text{\# of components may increase } x^c \text{ or } x^{c+1}} - \underbrace{P(G\setminus e,x)}_{\text{\# components stay smallest } x^c}$

COMPONENTS OF $G = c$

Proposition 8. *Let G be a graph on n vertices and with m edges, and t triangles. Then, the coefficient at x^{n-2} of $P(G,x)$ is*

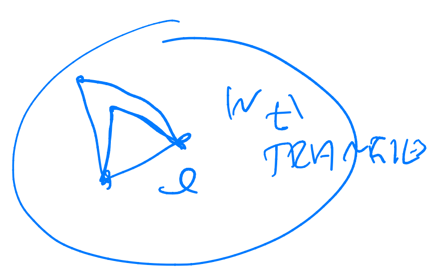
$$\binom{m}{2} - t\,.$$

12: Prove the proposition. Hint: Induction on m.

$m=0 \qquad P(\overline{K_n}, x) = x^n - 0x^{n-1} + 0 \cdot x^{n-2}$

$0 = \binom{m}{2} - t = \binom{0}{2} - 0 = 0$

$$P(G,x) = P(G-e,x) - P(G\setminus e,x)$$



x^{n-2} $\binom{m-1}{2} - (t - t')$

L AT x^{n-2} FOR $G \setminus e$

$|E(G \setminus e)|$ x^{n-2} → $m-1-t'$

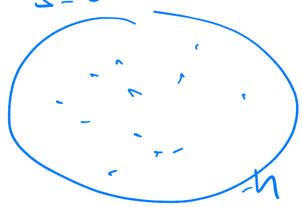
$$\binom{m-1}{2} - (t - \cancel{t'}) + m - 1 - \cancel{t'} = \boxed{\frac{(m-1)(m-2)}{2} + \frac{2(m-1)}{2}} - t$$

$$\binom{m}{2} - t$$

2 Expanding theorem

Let $c(G)$ the number of components of a graph G. For an arbitrary subset of edges $S \subseteq E(G)$ let $G[S]$ be the subgraph of G induced by S.

$S = \emptyset = (-1)^0 k^n$ $\quad S = 0$



Theorem 9. *For any graph G, it holds*

$$P(G,k) = \sum_{S \subseteq E(G)} (-1)^{|S|} k^{c(G[S])}. \tag{3}$$

In the proof we apply the complementary principle of inclusion–exclusion:

$$|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = \sum_{\mathcal{I} \subseteq \{1,2,\ldots,n\}} (-1)^{|\mathcal{I}|} |\bigcap_{i \in \mathcal{I}} A_i|, \tag{4}$$

where for $\mathcal{I} = \emptyset$ we assume the corresponding interesection is the union of all A_i.

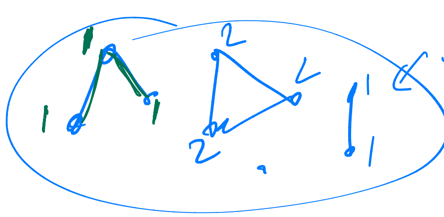
13: Prove the Theorem.

COLORINGS $V(G) \to \{1, \dots, k\}$ $\quad$ COLORING BAD FOR e

FOR $uv = e \in E(G)$ DENOTE A_e COLORINGS WHERE e IS MONOCHROMATIC

FIND $\quad$ NO BAD EDGE ⇒ "PROPER COLORINGS"

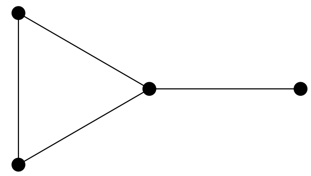
$$|A_1^c \cap A_2^c \cap \dots \cap A_m^c| = \sum_{S \subseteq \{1,2,\dots m\}} (-1)^{|S|} \left|\bigcap_{i \in I} A_i\right| \overset{?}{=} k^{c(G[S])}$$



2.1 The number of acyclic orientations of graph

Let $a(G)$ be the number of acyclic orientations of a graph G.

14: For the depicted graph G count $a(G)$, $P(G,x)$, and $P(G,-1)$



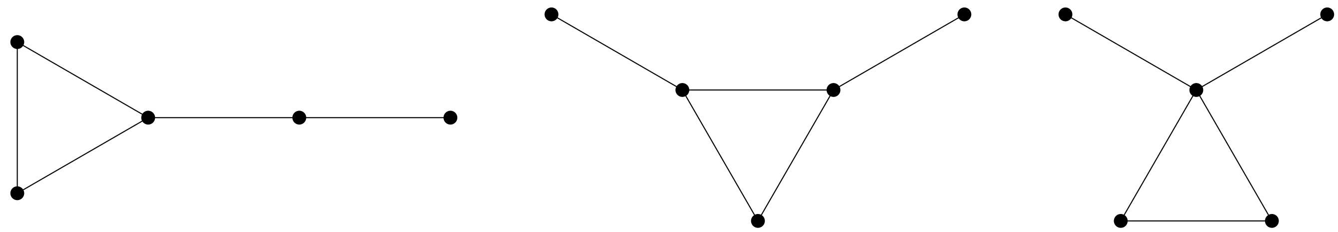
Proposition 10. *For an aribitraty graph G on n vertices, it holds*

$$P(G,-1) = (-1)^n a(G)\,.$$

15: Prove the proposition. Use induction on the number of edges.

3 Chromatically equivalent graphs

16: Find chromatic polynomials for the depicted graphs.



We say graphs G and H are *chromatically equivalent*, if they have a same chromatic polynomial, i.e. $P(G,x) = P(H,x)$.

In case that a graph G has no chromatically equivalent graphs, we say that G is a *chromatically unique* graph.

17: Find the chromatic polynomial of a cycle C_n

Proposition 11. *C_n is a chromatically unique graph.*

18: Prove the proposition.

One of the research directions on chromatic polynomial is to classify chromatically equivalent graphs, and also chromatically unique graphs.

3.1 Read's Conjecture

R. C. Read in 1968 proposed the following conjecture of unimodality of the coefficients of the chromatic polynomial.

Conjecture 12. *Let G be a graph and $P(G,x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x$ its chromatic polynomial. Then there exists an index k such that*

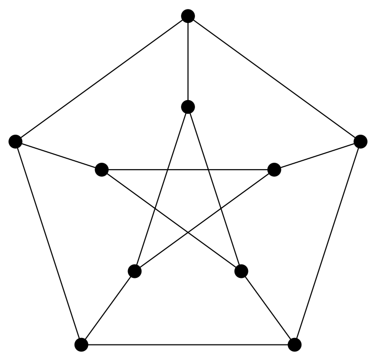
$$|a_n| \leq |a_{n-1}| \leq |a_{n-2}| \leq \cdots \leq |a_k| \geq |a_{k-1}| \geq \cdots \geq |a_2| \geq |a_1|.$$

We can see for example this on the Pteresen graph. Its chromatic polynomial is

$$x^{10} - 15x^9 + 105x^8 - 455x^7 + 1353x^6 - 2861x^5 + 4275x^4 - 4305x^3 + 2606x^2 - 704x.$$

The corresponding sequence is

$$1 < 15 < 105 < 455 < 1353 < 2861 < 4275 < 4305 > 2606 > 704\,.$$



Not so long ago the conjecture was solved by June Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, J. Amer. Math. Soc., **25** (2012) 907–927.